Stellar Populations

For many modern applications, one is not concerned with the evolution of a single star, but with an entire set of stars. There are a number of sophisticated computer codes that track this, but the calculations are actually fairly straightforward (and largely analytic).

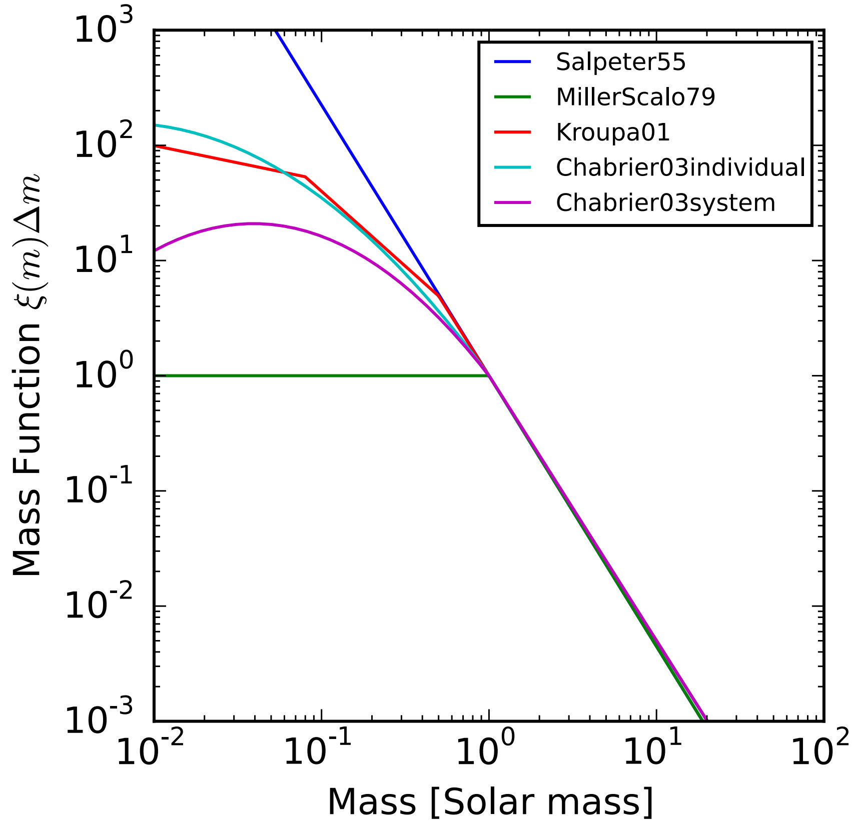
In order to make the mathematics a bit more straightforward, let's choose a fiducial mass, m_1, which can be the mass of the Sun. Such a star will have a mean main-sequence luminosity, ℓ_1, and a main-sequence lifetime of τ_1. We will then define the variable, m as the dimensionless mass of a star, *i.e.*, for a star of mass M, $m = M/m_1$.

First we need is an Initial Mass Function (IMF) of stars, which gives the number of stars born as a function of mass. The original IMF was that by Salpeter (1955) which was a simple power law,

$$\phi(m)dm = \phi_1 m^{-(1+x)} dm = \phi_1 m^{-s} dm \qquad (31.01)$$

with $x = 1.35$ (or $s = 2.35$), and limits between $0.1\,M_\odot$ and $100\ M_\odot$. Other commonly used IMFs include that by Miller & Scalo (1979), Kroupa *et al.* (1993), and Chabrier (2003). (Most of the differences are at the low-mass end of the function.) The constant ϕ_1 is simply there for normalization purposes, so that a star cluster (or galaxy) with total mass $\mathcal{M}_0$ satifies

$$\mathcal{M}_0 = \mathcal{M}_0 \int_{m_{\min}}^{m_{\max}} m\,\phi(m)\,dm \qquad (31.02)$$



The second piece of information we need is the mass-luminosity relation for main-sequence stars. As we have seen, this is a roughly a power-law, with a slope of $\alpha \approx 3.88$ at the faint end, and a slightly flatter relation at higher masses. For simplicity, we'll use a single power law connecting mass to luminosity on the main sequence

$$\ell_d = \ell_1 m^\alpha \tag{31.03}$$

where the subscript d represents the luminosities of dwarfs (*i.e.,* main sequence stars), and $\alpha \sim 3.5$.

The third item needed is the length of time stars spend on the main sequence. The lifetime of a star is simply proportional to the energy available to the star divided by rate at which the star is emitting its energy, *i.e.,* its luminosity. Since the available energy is proportional to the available mass, and the luminosity

is proportional to mass (though the mass-luminosity relation),

$$\tau \propto \frac{m}{\ell_d} \propto \left(\frac{m}{m^\alpha}\right) \propto m^{1-\alpha} \implies \tau = \tau_1 m^{1-\alpha} \tag{31.04}$$

where τ_1 is the lifetime of the fiducial star with $m = m_1$. Alternatively, this equation can be inverted. After a time t, the turnoff mass of a single-age stellar population will be

$$m_{tn} \propto t_{tn}^{1/(1-\alpha)} \implies m_{tn} = \left(\frac{t}{\tau_1}\right)^{1/(1-\alpha)} \tag{31.05}$$

The final pieces of information that are needed are the lifetime of a typical post-main sequence (giant) star (τ_g), the average luminosity of a giant star, ℓ_g, and the mass of a remnant star (typically a white dwarf, w). All of these can be derived (approximately) from models of stellar evolution. Note that $\tau_g \ll \tau_d$, so the stellar main-sequence lifetime is also roughly the star's total lifetime.

We will now consider the evolution of a set of stars all born (with the same metallicity) at the same time. This is called a Simple Stellar Population (SSP). The properties of more complicated systems can be inferred from the summation of several SSPs.

Luminosity Evolution

Let's first calculate the luminosity evolution of a cluster (or galaxy) of stars with total mass $\mathcal{M}_0$ all born at the same time. The total luminosity of main sequence stars is easy to compute: all we have to do is sum up the luminosity of all stars on the main sequence. The lower end of the sum is the minimum mass of an energy-generating star; the upper end is defined by the main sequence turnoff. In other words,

$$\begin{aligned}
\mathcal{L}_D(t) &= \int_{m_L}^{m_{tn}} \mathcal{M}_0\, \phi(m)\, \ell_d(m)\, dm \\
&= \int_{m_L}^{m_{tn}} \mathcal{M}_0 \;\cdot\; \phi_1 m^{-(1+x)} \cdot \ell_1 m^{\alpha}\, dm \\
&= \mathcal{M}_0 \phi_1 \ell_1 \int_{m_L}^{m_{tn}} m^{\alpha-1-x}\, dm \\
&= \frac{\mathcal{M}_0 \phi_1 \ell_1}{\alpha - x} \left\{ m_{tn}^{\alpha-x} - m_L^{\alpha-x} \right\} \\
&= \frac{\mathcal{M}_0 \phi_1 \ell_1}{\alpha - x} \left\{ \left(\frac{t}{\tau_1}\right)^{\frac{\alpha-x}{1-\alpha}} - \left(\frac{t_L}{\tau_1}\right)^{\frac{\alpha-x}{1-\alpha}} \right\} \qquad (31.06)
\end{aligned}$$

Note that since the exponent is negative, and low mass stars live (essentially) forever, the last term in the above equation is negligible. Thus, the total luminosity of dwarf stars is

$$\mathcal{L}_D(t) \approx \frac{\mathcal{M}_0 \phi_1 \ell_1}{\alpha - x} \left(\frac{t}{\tau_1}\right)^{(\alpha-x)/(1-\alpha)} \qquad (31.07)$$

Calculating the total luminosity of giant stars is equally simple. First, note that the timescale for giant branch evolution is much faster than that for main sequence evolution. Thus, the key is to estimate the number of stars currently turning into giants; when this number is multiplied by the length of time a typical star remains a giant, and the mean luminosity of the star, the result is the total giant star luminosity. Now consider: the rate at which main sequence stars turn into giants is defined by how many stars are at the main sequence turnoff, and how much of the main sequence is eaten away per unit time interval. According to our definition of the initial mass function, the number of stars at the main-sequence turnoff is

$$\frac{dN(m)}{dm} = \phi(m) = M_0 \phi_1 m^{-(1+x)} \tag{31.08}$$

and the number of stars turning into red giants during a time Δt is

$$N_g = \frac{dN}{dt} \Delta t = \frac{dN}{dm} \cdot \frac{dm}{dt} \cdot \Delta t \tag{31.09}$$

so the total luminosity of giant stars is

$$\begin{aligned} \mathcal{L}_G &= \phi(m_{tn}) \cdot \frac{dm_{tn}}{dt} \cdot \tau_g \cdot \ell_g \\ &= \mathcal{M}_0 \phi_1 m_{tn}^{-(1+x)} \frac{dm_{tn}}{dt} \ell_g \, \tau_g \end{aligned} \tag{31.10}$$

If we now substitute for t for m using (31.05) take the derivative, and combine terms, the result is

$$\mathcal{L}_G = \frac{\mathcal{M}_0 \phi_1 \, \ell_g \, \tau_g}{\tau_1 (\alpha - 1)} \left(\frac{t}{\tau_1} \right)^{\frac{\alpha - x - 1}{1 - \alpha}} \tag{31.11}$$

Thus, the total luminosity of the stellar system, as a function of time, is

$$\mathcal{L}_T = \mathcal{L}_D + \mathcal{L}_G = \frac{\mathcal{M}_0 \phi_1 \ell_1}{\alpha - x} \left(\frac{t}{\tau_1} \right)^{\frac{\alpha - x}{1-\alpha}} + \frac{\mathcal{M}_0 \phi_1 \, \ell_g \, \tau_g}{\tau_1 (\alpha - 1)} \left(\frac{t}{\tau_1} \right)^{\frac{\alpha - x - 1}{1-\alpha}} \tag{31.12}$$

This can be simplified a bit if we define $G(t)$ as the ratio of giant star luminosity to dwarf star luminosity, *i.e.*,

$$G(t) = \frac{\mathcal{L}_G}{\mathcal{L}_D} = \frac{(\alpha - x) \, \ell_g \, \tau_g}{(\alpha - 1) \, \ell_1 \, \tau_1} \left(\frac{t}{\tau_1} \right)^{1/\alpha - 1} \tag{31.13}$$

Since $\alpha > 1$, the exponent of time in (31.13) is significantly less than one. Hence writing the expression for $G(t)$ in this way illustrates that the variable is only a weak function of time. This expression can then be further simplied by using (31.03) and (31.05)

$$\left(\frac{t}{\tau_1} \right)^{\frac{1}{\alpha - 1}} = \left(\frac{t}{\tau_1} \right)^{\frac{\alpha}{\alpha - 1}} \left(\frac{t}{\tau_1} \right)^{-\frac{\alpha - 1}{\alpha - 1}} = m^{-\alpha} \left(\frac{\tau_1}{t} \right) = \frac{\ell_1 \tau_1}{\ell_{tn} t} \tag{31.14}$$

Thus

$$G(t) = \frac{\alpha - x}{\alpha - 1} \left\{ \frac{\ell_g \, \tau_g}{\ell_{tn} \, t} \right\} \tag{31.15}$$

Note that the first part of this equation is very close to one. Furthermore, the term in brackets is simply the ratio of the total energy generated by the star on the giant branch to the total energy the star generated on the main sequence. Since stars turn off the main sequence when $\sim 10\%$ of their total fuel is gone, but eventually burn $\sim 70\%$ of their fuel, $G(t) \sim 6$. (From an observers point of view, it's a bit more complicated, since most

of the light a giant star produces is red, while the main sequence light may be blue. Thus, the exact value of $G(t)$ depends on the wavelength of observation: in the blue, $G(t) \sim 1$, but in the red, $G(t) > 10$.) Using this notation, the luminosity of a stellar population, as a function of time, is

$$\begin{aligned}\mathcal{L}_T(t) &= \mathcal{L}_D(t)\,\{1 + G(t)\} \\ &= \frac{\mathcal{M}_0 \phi_1 \ell_1}{\alpha - x}\,\{1 + G(t)\} \left(\frac{t}{\tau_1}\right)^{\frac{\alpha - x}{1-\alpha}} \end{aligned} \tag{31.16}$$

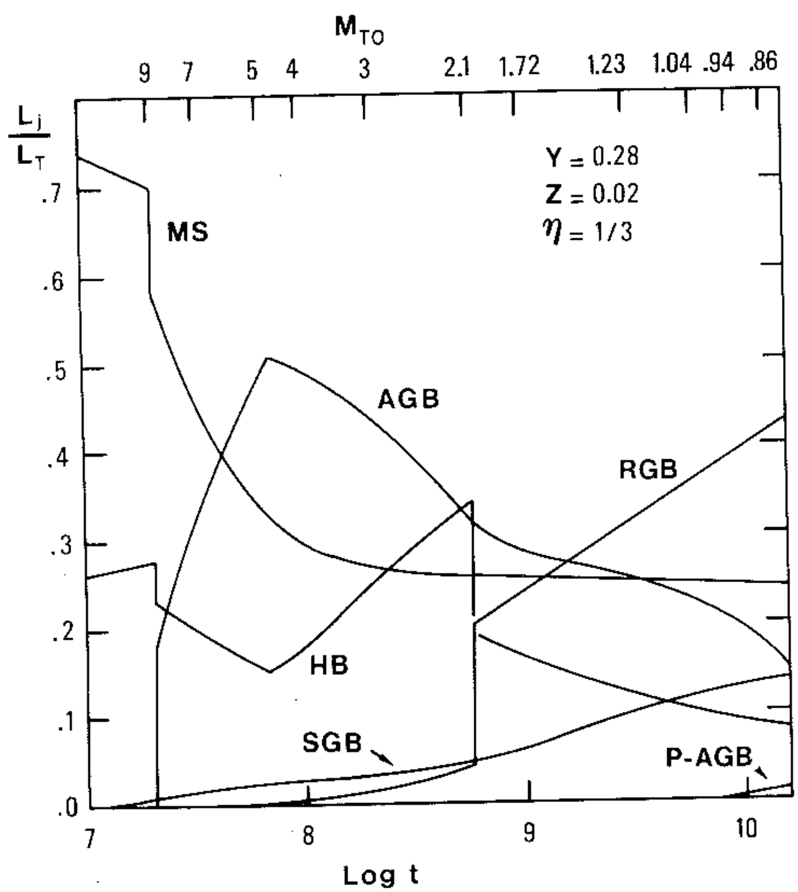


Figure 5. The relative contribution of the various evolutionary stages to the integrated bolometric luminosity of a simple stellar population as a function of age (lower scale) and turnoff mass (upper scale). The various evolutionary stages are defined in the text, composition and mass loss parameter are indicated, and s = 2.35.

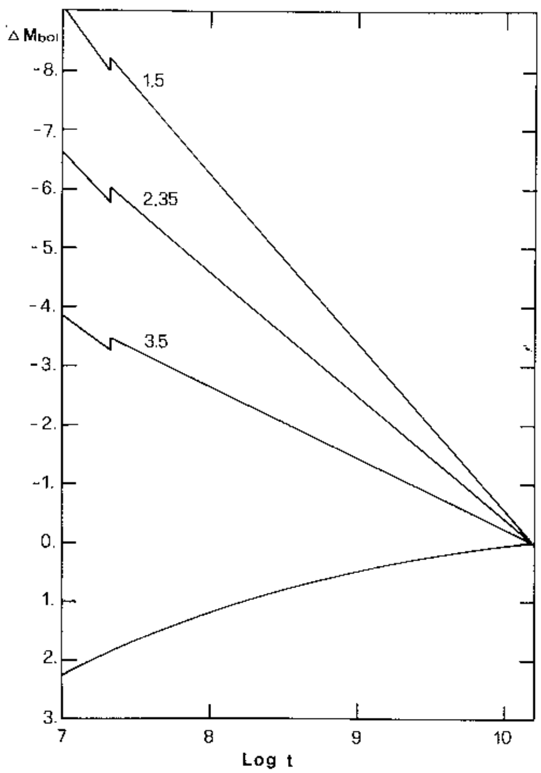


Figure 4. The luminosity evolution of SSP's for three choices of the IMF slope s, normalized at $\Delta M_{bol} = 0$ for t = 15 Gyrs. The lower line shows the luminosity evolution of a population with constant star formation rate, s = 2.35, and upper IMF cutoff at 50 $M_{\odot}$.

Photometric Mass-to-Light Ratio

In addition to a population's total luminosity, there are several other quantities which can be computed from first principles. First is the photometric mass-to-light ratio. Recall the total luminosity of main sequence stars in a stellar population is

$$\mathcal{L}_D = \frac{M_0 \phi_1 \ell_1}{\alpha - x} \left\{ m_{tn}^{\alpha - x} - m_L^{\alpha - x} \right\} \tag{31.06}$$

where m_{tn} is the turnoff mass of the main sequence. As stated previously, $\alpha - x$ is usually greater than zero, so the last term of this equation is negligible; stars at the low end of the mass sequence do not contribute significantly to the system's luminosity.

On the other hand, low mass stars can be an important part of the cluster's mass. From (31.02), the visible mass of stars in a galaxy is given by

$$\begin{aligned} \mathcal{M} = M_0 \int_{m_L}^{m_U} \phi(m) m \, dm &= M_0 \int_{m_L}^{m_U} \phi_1 m^{-x} dm \\ &= \frac{M_0 \phi_1}{1 - x} \left\{ m_{tn}^{1-x} - m_L^{1-x} \right\} \end{aligned} \tag{31.17}$$

(neglecting the mass in invisible stellar remnants). If $x > 1$ (as is normally assumed), the exponents in (31.17) are less than zero, and the value of $\mathcal{M}$ is dominated by the last term in the equation. Physically, this means that most of the stellar mass of a population resides in low mass stars.

The combination of (31.06) and (31.17) means that it is virtually impossible to determine a population's photometric mass-to-light ratio from observations alone. One can always drive up $\mathcal{M}/\mathcal{L}$ by postulating the existence of low-mass stars which add mass (via 31.17), without adding luminosity (31.06).

Stellar Evolutionary Flux

An interesting property associated with stellar populations is the stellar evolutionary flux, *i.e.,* the number of stars passing through any (post main-sequence) phase of evolution at a given time. Just as the amount of water flowing down a river is controlled by the spillway of a dam, the number of stars evolving through various stages of evolution is controlled by the rate at which these stars move off the main sequence. (All other rates are much faster than this.) As we have seen previously, this rate is

$$N_{tn} = M_0 \phi(m_{tn}) \frac{dm_{tn}}{dt} = M_0 \phi_1 m^{-(1+x)} \frac{dm_{tn}}{dt} \tag{31.18}$$

From (31.05)

$$m = \left(\frac{t}{\tau_1}\right)^{1/1-\alpha} \implies \frac{dm_{tn}}{dt} = \frac{1}{\tau_1(1-\alpha)} \left(\frac{t}{\tau_1}\right)^{\alpha/1-\alpha} \tag{31.19}$$

so the stellar evolutionary flux is

$$N_{tn} = \frac{M_0 \phi_1}{\tau_1(1-\alpha)} \left(\frac{t}{\tau_1}\right)^{\frac{x-1-\alpha}{\alpha-1}} \tag{31.20}$$

Naturally, the number of stars evolving through any phase of evolution in a galaxy is proportional to the number of stars in the galaxy. The best way to take this dependence out is to normalize the evolutionary flux to the size of the galaxy. This cannot be done with mass, since we do not know the total number of stars that are present in the system. However, we do know the population's luminosity, which is defined in (31.16). So, when we normalize (31.20) to this luminosity, we arrive at the population's **luminosity-specific stellar evolutionary flux**. After a bit of math, this quantity comes out to be

$$B = \frac{N_{tn}}{L_t} = \frac{\alpha - x}{(\alpha-1)(1+G(t))\ell_1 \tau_1} \left(\frac{t}{\tau_1}\right)^{\frac{1}{\alpha-1}} \tag{31.21}$$

Note that the exponent of time is less than 1; thus B is rather insensitive to the age of the stellar population. (The difference between the stellar evolutionary flux of a 7 Gyr stellar population and a 12 Gyr population is less than a 25%.) Similarly, the specific stellar evolutionary flux is not very sensitive to the exact value of x.

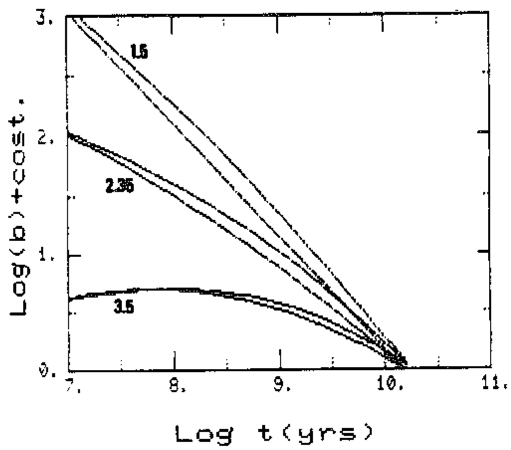
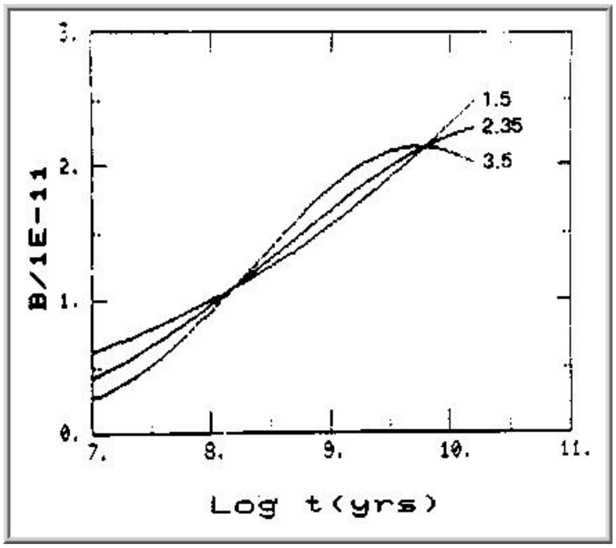
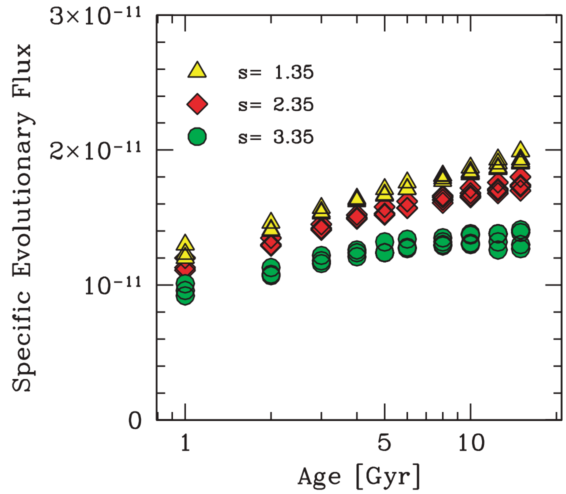


Figure 2. The evolutionary flux b(t) from Eq. (3) for three choices of the IMF slope s, normalized at b = 1 for t = 15 Gyrs. For each value of s the upper line gives the death rate, $b(t - t_{PMS})$.

Detailed calculations show that the luminosity-specific stellar evolutionary flux is $B \approx 2 \times 10^{-11}$ stars yr^{-1} $L_{\odot}^{-1}$ for all old (or moderately old) stellar populations. This makes is easy to predict the number of any post-main sequence type of star present in a galaxy: since you know B, and you can measure the luminosity of the galaxy, all you need is the lifetime of the phase in question. For example, the lifetime of a planetary nebula is $\tau \sim 25,000$ years. If a galaxy is observed to have an absolute luminosity of $\mathcal{L} = 10^{11}\mathcal{L}_{\odot}$, then the number of PN in the galaxy is $N(PN) \approx B \cdot \mathcal{L} \cdot \tau = 50,000$.

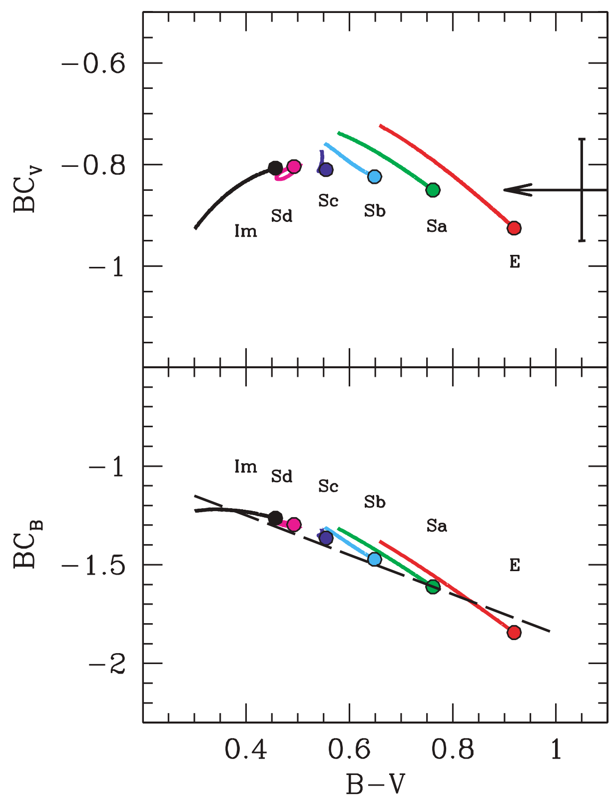




The plot above shows how a population's bolometric-luminosity specific stellar evolutionary flux changes with the systems age and IMF. Note that the x-axis (population age) is a log quantity, while the y-axis is linear. All old ($>$ 3 Gyr) stellar population evolve at rate of $\sim 2 \times 10^{-11}$ stars per year per solar luminosity, independent of age, IMF, or metallicity.

[Renzini & Buzzoni 1986, in *Spectral Evolution of Galaxies,* p. 195]

Two notes. First, the theoretical calculation of the luminosity-specific stellar evolutionary flux are based on *bolometric* luminosity. One rarely has access to the entire electromagnetic spectrum of a system of stars – most often the amount of luminosity surveyed in a star system is given in a specific filter, such as V or B. Fortunately, the conversion between V-luminosity and bolometric luminosity is very insensitive to the details of the stellar population. So, a bolometric correction of ~ -0.85 can convert V to bolometric magnitude.



Second: a common idea these days is that all (or many) planetary nebulae are formed out of binary star interactions. If so, then the previous calculation may not work, since it does not take into account binary-star interactions. (Interestingly, however, the estimate does appear to come close to predicting PN numbers.) Similarly, if not all stars evolve through a given stage, then the calculation may be in error.

Mass Loss from Stars

Another useful quantity to know is the rate of mass loss from stars as a function of time. As before, the key to calculating this is to realize that almost all the mass lost from stars comes during the post main-sequence phase of evolution. Thus, the rate at which a population of stars loses mass is just by the rate at which stars move off the main sequence times the amount of mass each star loses. If m is the initial mass of the star, and w the mass of the stellar remnant (*i.e.*, white dwarf), then the mass ejection rate is

$$E(t) = N_{tn}(m_{tn} - w) = M_0 \phi(m_{tn}) \frac{dm_{tn}}{dt}(m_{tn} - w) \tag{31.22}$$

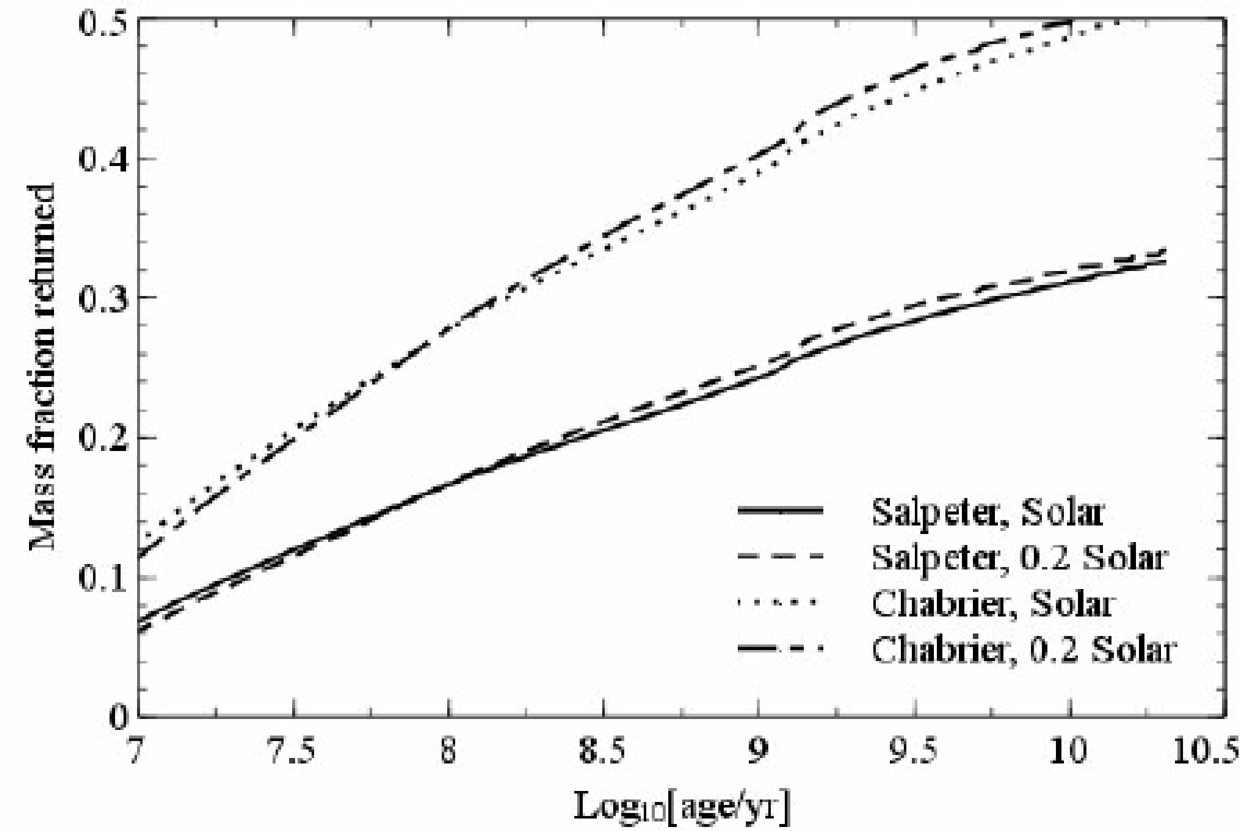
After performing the same substitutions as was done for the calculation of stellar evolutionary flux, this simplifies to

$$E(t) = \frac{M_0 \phi_1}{\tau_1 (1 - \alpha)}(m_{tn} - w) \left(\frac{t}{\tau_1}\right)^{\frac{x-1-\alpha}{\alpha-1}} \tag{31.23}$$

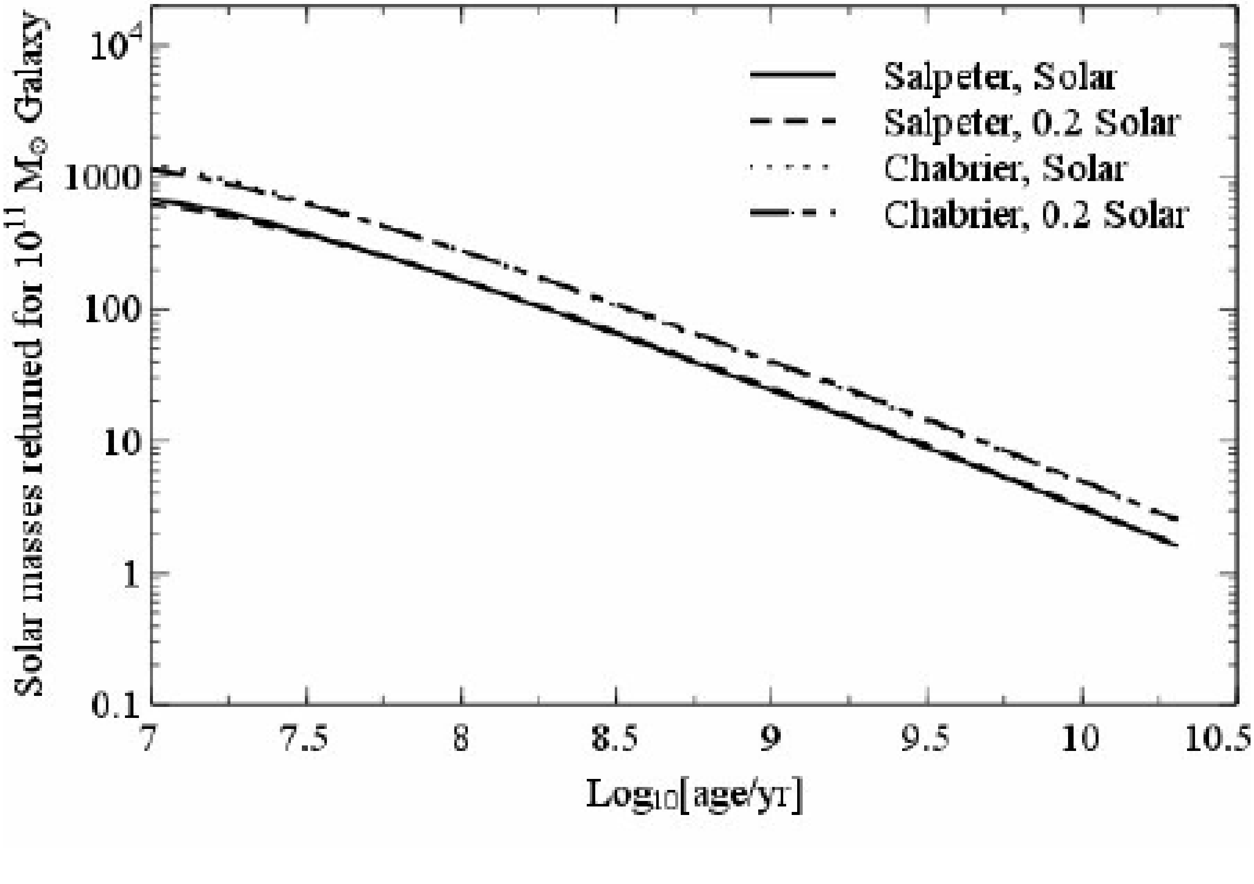
Once again, if we normalize to luminosity via (31.16), we can derive the luminosity specific mass loss rate

$$\frac{E(t)}{\mathcal{L}_t} = \frac{\alpha - x}{1 + G(t)} \frac{(m_{tn} - w)}{\alpha - 1} \frac{1}{\ell_{tn} t} \tag{31.24}$$

Numerically, this works out to a mass loss rate of $\sim 0.02 \mathcal{M}_\odot$ per Gigayear per unit solar luminosity for an old (10^{10} yr) stellar population. When integrated over the lifetime of a galaxy, the result is that $\sim 15\%$ of the initial stellar mass will be lost during a Hubble time.



Mass fraction returned
Log10[age/yr]
Salpeter, Solar
Salpeter, 0.2 Solar
Chabrier, Solar
Chabrier, 0.2 Solar



Solar masses returned for 10^{11} $M_\odot$ Galaxy
Log10[age/yr]
Salpeter, Solar
Salpeter, 0.2 Solar
Chabrier, Solar
Chabrier, 0.2 Solar

More Complex Systems

Real galaxies usually contain multiple stellar populations, each with its own mass, age, and metallicity. Computing the behavior of such a system simply means summing variations SSP components. It would, however, be useful to some zeroth order approximation to know the mix of stellar populations within a galaxy.

A common way to do this is to simply assume that the star-formation rate of a galaxy declines exponentially with time, with some decay rate τ, *i.e.,*

$$\mathcal{M}(t) \propto e^{-t/\tau} \tag{31.25}$$

Irregular and late-type spiral galaxies are assigned large values of τ, suggesting an almost constant star-formation rate over the history of the universe. Elliptical galaxies have small values of τ, thus reflecting the fact that their star-formation has e-folded away long ago. Note however, that, many/most galaxies likely undergo a succession of discrete starbursts, with low-level star formation occuring in between times of extreme activity. So these models may be of only moderate use.

Problems in Photometric Evolution Calculations

The above analytic solutions are only useful for guidance. To compare with observations, predictions must be made in specific bandpasses, or for specific absorption lines. This requires numerical calculations which include

1) sets of stellar isochrones, detailing the precise number of stars at any position in the HR diagram as a function of age.

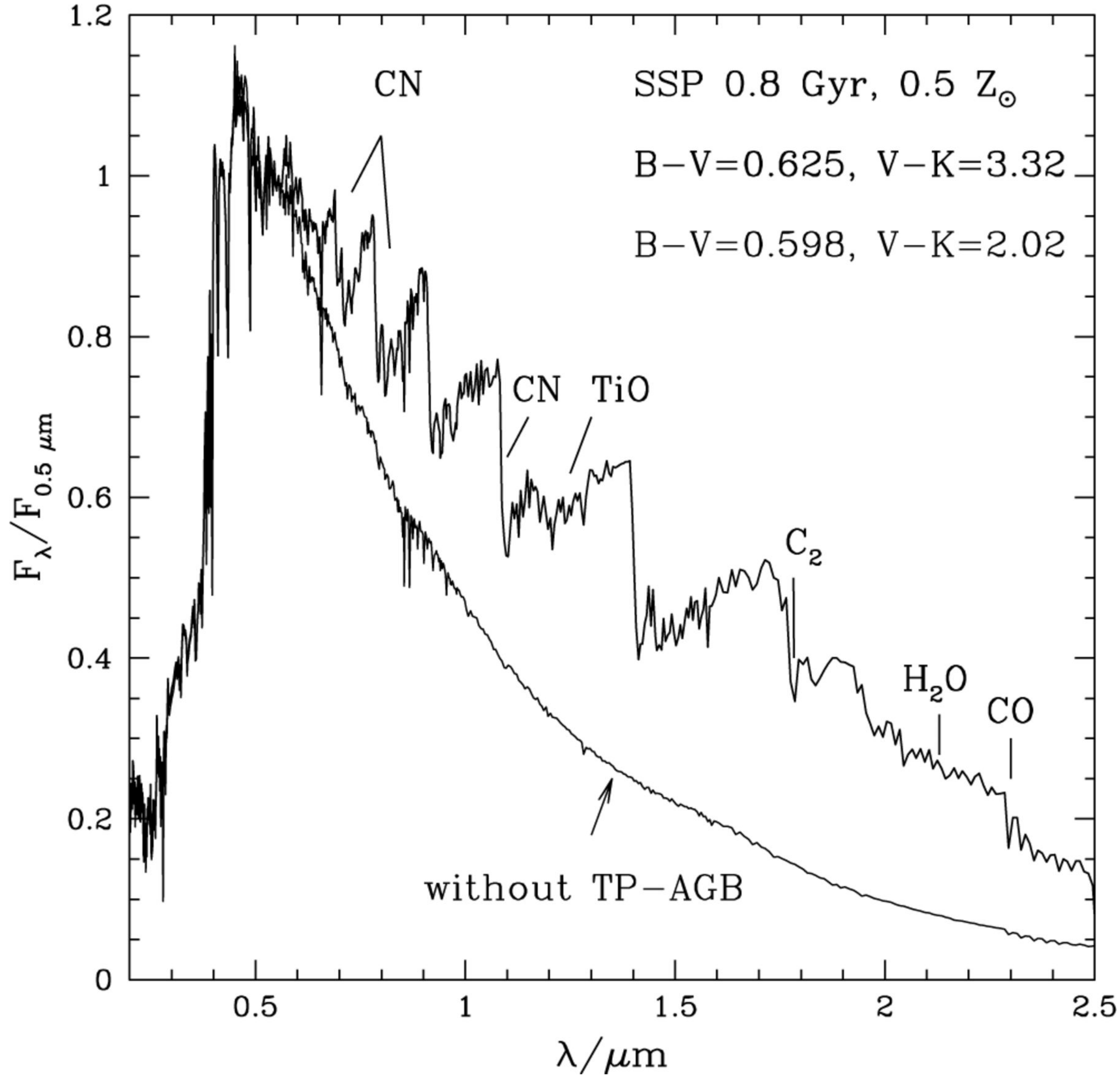
2) sets of model stellar atmospheres, giving the emitted spectrum (or broadband color) of each star, as a function of its temperature, surface gravity, and metallicity.

3) an understanding of various non-stellar processes that may effect the emergent spectrum from galaxies. Such effects include internal extinction due to dust, the presence of emission-lines from ionized gas, and (for high-redshift objects) absorption due to the intervening intergalactic medium (*i.e.,* the Lyα forest).

Today, the most controversial parts of these codes include their treatment of thermally pulsing AGB stars, and the proper modelling of the horizontal branch and post-AGB stars. The former is important since it involves extremely luminous stars, which, due to their mass loss, may be obscured by circumstellar extinction. The latter is important for modeling old stellar systems which have very few blue stars. For these objects, a small change in the morphology of the horizontal branch (*i.e.,* the ratio of red to blue objects), or the number of slow-evolving post-AGB stars can change the system's UV color dramatically.

Finally, most (but not all) stellar population codes neglect binary stars and binary evolution. In part, this is because modeling the flux distribution from a binary population involves *lots* of

additional free parameters. But binary evolution probably is important!



Chemical Evolution

Another topic related to the photometric evolution of a stellar population is the chemical evolution produced by a set of stars. The topic connects the evolution of single stars to the evolution of a galaxy.

To start, let's consider the types of parameters and variables that are involved. First, there are the global variables, all of which are a function of time.

M_g: Total mass of interstellar gas
M_s: Total mass of stars
M_w: Total mass of stellar remnants (white dwarfs)
M_t: Total mass of the system
E: the rate of mass ejection from stars
E_Z: the rate of metal ejection from stars
W: the creation rate of stellar remnants.

Naturally, $M_t = M_g + M_s + M_w$.

Next, there are the global parameters of the model, which the investigator specifies. Again, all can be a function of time.

Ψ: Rate of star formation
f: Rate of infall or outflow of material from the system
Z_f: Metal abundance of the infall (or outflow) material
$\phi(m)$: the Initial Mass Function

As in the equations of photometric evolution, the IMF should be normalized so that the total mass is one, as in (31.02).

Finally, you need four variables which come from stellar evolution

w:	the mass of a stellar remnant
τ_m:	the main-sequence lifetime of a star
m_{tn}:	the turnoff mass of a population with $t = \tau$
p_z:	the stellar recyclable mass fraction that is converted to metal z and then ejected into space.

Given the above variables and parameters, the goal is to derive $Z(t)$, the fraction of metals (individually, or as a group) in the interstellar medium as a function of time.

Equations of Chemical Evolution

There are five coupled differential equations which describe the chemical evolution of a system.

$$\frac{dM_t}{dt} = f \tag{32.01}$$

$$\frac{dM_s}{dt} = \Psi - E - W \tag{32.02}$$

$$\frac{dM_g}{dt} = -\Psi + E + f \tag{32.03}$$

$$\frac{dM_w}{dt} = W \tag{32.04}$$

$$\frac{d(ZM_g)}{dt} = -Z\Psi + E_Z + Z_f f \tag{32.05}$$

The equations are fairly simple to understand. Equation (32.01) is simple mass conservation. Equations (32.02), (32.03), and (32.04) keep track of the amount of mass that gets locked up in stars (or released into the ISM). Equation (32.05) is the most complex, as it describes how the metallicity of the interstellar medium changes with time. The first term of (32.05) refers to the amount of ISM metals that becomes locked up into stars, the second term gives the amount of metals being released by stars, and the third represents the amount of metals being brought in (or lost) from outside.

Although the list of variables mentioned above is formidable, not all the variables are independent. Consider E, the mass ejection rate from stars. Since mass loss only occurs during post-main sequence evolution, the rate of mass loss is related to the number of stars evolving off the main sequence at any time. If the galaxy

consisted only of a single population of stars, this rate would be given by

$$E(t) = N_{tn}(m_{tn} - w) = \mathcal{M}_0 \phi(m_{tn}) \frac{dm_{tn}}{dt}(m_{tn} - w) \quad (31.22)$$

However, for a galaxy with on-going star formation, the calculation of mass ejection rate must count the main-sequence turnoff stars of all stellar ages. Thus the total amount of ISM returned from stars at time t is

$$E = \int_{m_{tn}}^{m_u} (m - w)\Psi(t - \tau_m)\phi(m, t - \tau_m)dm \quad (32.06)$$

where m_u is the upper mass limit of the stellar IMF, and m_{tn}, the turnoff mass at time t. Similarly, the equation for the total mass of remnants formed is

$$W = \int_{m_{tn}}^{m_u} w\Psi(t - \tau_m)\, \phi(m, t - \tau_m)dm \quad (32.07)$$

The equation for E_Z is a bit more complicated since it has two terms: one to represent the amount of *new* metals created by a star and released during mass loss, and a second to represent the amount of metals that were lost from the ISM when the star formed, but are now being re-released. Mathematically, this is

$$E_Z = \int_{m_{tn}}^{m_u} mp_z \Psi(t - \tau_m)\, \phi(m, t - \tau_m)dm + \int_{m_{tn}}^{m_u} (m - w - mp_z)Z(t - \tau_m)\Psi(t - \tau_m)\, \phi(m, t - \tau_m)dm \quad (32.08)$$

Finally, there is an equation of metal conservation. If $\bar{Z}_s$ is the average metal content in stars, then the total amount of metals

produced in a galaxy over a Hubble time is

$$\bar{Z}_s M_s + Z M_g = \int_0^t \int_{m_{tn}}^{m_u} m p_z \Psi(t' - \tau_m) \phi(m, t' - \tau_m) dt' dm \tag{32.09}$$

Primary and Secondary Elements

The above equations assume that p_z, the fraction of a star which is converted into metals, is independent of the initial metallicity of the star. In other words, it assumes that the metal under consideration is a primary element. However, some elements can only be made if another element already exists. For example, to make nitrogen (via the CNO cycle), the star must already have some carbon. Thus, the mass ejection rate of a secondary element, X, is

$$E_X = \int_{m_{tn}}^{m_u} m p_X Z(t-\tau_m)\Psi(t-\tau_m)\,\phi(m,t-\tau_m)dm + \int_{m_{tn}}^{m_u} (m - w - m p_X) X(t-\tau_m)\Psi(t-\tau_m)\,\phi(m,t-\tau_m)dm \tag{32.10}$$

Note that this is similar to the equation for E_Z, in that it has two terms: the creation term and the recycle term. However, in this case, the creation term depends on the prior abundance of Z.

Analytic Approximation to Chemical Evolution

Obviously, solving the above coupled differential equations with their four free parameters (Ψ, ϕ, f, and Z_f) is a non-trivial numerical problem. However, the problem can be greatly simplified if you make two approximations.

The first approximation to make is to say that the initial mass function of stars is independent of time. That is, $\phi(m, t) = \phi(m)$. Since little is known about how the IMF changes as a function of galactic conditions, this may, or may not, be a good assumption. (The prevailing theory is that the IMF for a system with *no* metals is heavily biased towards extremely massive stars, but once the first bits of metals get introduced into the ISM, this bias goes away.)

The second approximation is call the *instantaneous recycling* approximation and it is a bit tricker. The approximation says that there are two types of stars in a galaxy: those that live forever, and those that evolve and die instantaneously. Although this sounds like a poor assumption, it's not as bad as it first appears. Recall that the timescales for stellar evolution:

Main Sequence Lifetimes

Spectral Type	Mass $(\mathcal{M}/\mathcal{M}_\odot)$	Luminosity $(\mathcal{L}/\mathcal{L}_\odot)$	Lifetime (years)
O5 V	60	7.9×10^5	5.5×10^5
B0 V	18	5.2×10^4	2.4×10^6
B5 V	6	820	5.2×10^7
A0 V	3	54	3.9×10^8
F0 V	1.5	6.5	1.8×10^9
G0 V	1.1	1.5	5.1×10^9
K0 V	0.8	0.42	1.4×10^{10}
M0 V	0.5	0.077	4.8×10^{10}
M5 V	0.2	0.011	1.4×10^{11}

Note the values. Stars with $\mathcal{M} > 5\mathcal{M}_\odot$ evolve in less than 10^8 years, which, in cosmological terms, is almost instantaneously. On the other hand, stars with mass less than about $1\mathcal{M}_\odot$ live forever. So the approximation only breaks down for a limited mass range.

Let's choose m_1 to be the dividing line between stars that live forever, and stars that evolve instantaneously. Let's also define three new quantities, the **Return fraction** of gas

$$R = \int_{m_1}^{\infty} (m - w)\phi(m)dm \tag{32.11}$$

the **Baryonic Dark Matter fraction**

$$D = \int_{m_1}^{\infty} w\phi(m)dm \tag{32.12}$$

and the **Net Yield** (of element i)

$$y_i = \frac{1}{1-R}\int_{m_1}^{\infty} mp_z\phi(m)dm \tag{32.13}$$

In words, R is the amount of mass a generation of stars puts back into the ISM, D is the amount of mass a generation of stars turns into stellar remnants, and y_i is the fraction of metal i produced by stars for every $1\mathcal{M}_\odot$ of material locked up into stars or remnants. The importance of these three quantities is that each depend only on the IMF. If we assume some universal form for the IMF, then R, D, and y_i are constants that depend only on stellar evolution. In other words, they are known quantities.

Now, let's take another look at the equations for E, W, and E_Z. If we assume $\phi(m)$ is independent of t and use the instantaneous recycling approximation, then

$$\begin{aligned} E &= \int_{m_{tn}}^{m_u}(m-w)\Psi(t-\tau_m)\phi(m,t-\tau_m)dm \\ &= \Psi(t)\int_{m_1}^{m_u}(m-w)\phi(m)dm = R\Psi \end{aligned} \tag{32.14}$$

Similarly, the equation for stellar remnants becomes

$$\begin{aligned} W &= \int_{m_{tn}}^{m_u} w\Psi(t-\tau_m)\phi(m,t-\tau_m)dm \\ &= \Psi(t)\int_{m_1}^{m_u} w\phi(m)dm = D\Psi \end{aligned} \tag{32.15}$$

and, after a bit of math,

$$E_Z = \Psi\left\{ZR + y_z(1-R)\right\} \tag{32.16}$$

With our two assumptions, the equations of chemical evolution become

$$\frac{d\mathcal{M}_t}{dt} = f \tag{32.17}$$

$$\frac{d\mathcal{M}_s}{dt} = (1 - R - D)\Psi \tag{32.18}$$

$$\frac{d\mathcal{M}_g}{dt} = -(1 - R)\Psi + f \tag{32.19}$$

$$\frac{d\mathcal{M}_w}{dt} = D\Psi \tag{32.20}$$

$$\frac{d(Z\mathcal{M}_g)}{dt} = -Z\Psi(1 - R) + y_z\Psi(1 - R) + Z_f f \tag{32.21}$$

Actually (32.21) can be further simplified by noting that

$$\frac{d(Z\mathcal{M}_g)}{dt} = Z\frac{d\mathcal{M}_g}{dt} + \mathcal{M}_g\frac{dZ}{dt} \tag{32.22}$$

Substituting (32.19) for dM_g/dt then yields

$$\mathcal{M}_g\frac{dZ}{dt} = y_z\Psi(1 - R) + (Z_f - Z)f \tag{32.23}$$

For secondary elements, the ejection rate from (32.10) becomes

$$\begin{aligned} E_X &= \Psi Z(1 - R)y_z + RX\Psi - y_z(1 - R)X\Psi \\ &= \Psi(1 - R)y_z(Z - X) + \Psi XR \end{aligned} \tag{32.24}$$

(In many cases, $X \ll Z$, so to simplify things, you can often get away with $Z - X \approx Z$.) Metal conservation for secondary elements is therefore

$$\frac{d(X\mathcal{M}_g)}{dt} = -X\Psi + \Psi(1 - R)(Z - X)y_x + RX\Psi + X_f f \tag{32.25}$$

But since

$$\frac{d(X\mathcal{M}_g)}{dt} = X\frac{d\mathcal{M}_g}{dt} + \mathcal{M}_g\frac{dX}{dt}$$

$$= -X(1-R)\Psi + Xf + \mathcal{M}_g\frac{dX}{dt} \tag{32.26}$$

some of the terms cancel, so

$$\mathcal{M}_g\frac{dX}{dt} = \Psi(1-R)(Z-X)y_x + (X_f - X)f \tag{32.27}$$

Estimating the Stellar Yields

Models of stellar evolution and supernova nucleosynthesis can predict the amount of metals produced by a stellar population. But is there a way to empirically check the results of these (purely theoretical) calculations?

Let's start with the equation for the total amount of metals produced over a systems lifetime. Using the instantaneous recycling approximation,

$$\begin{aligned}\bar{Z}_s(\mathcal{M}_s + \mathcal{M}_w) + Z\mathcal{M}_g &= \int_0^t \int_{m_{tn}}^{m_u} m p_z \Psi(t' - \tau_m)\phi(m, t' - \tau_m) dt' dm \\ &= \int_0^t \Psi(t) dt \int_{m_1}^{\infty} m p_z \phi(m) dm \\ &= (1 - R)\, y_z \Psi_T \qquad (32.28)\end{aligned}$$

where Ψ_T is the total amount of star formation over the history of the galaxy. Now that we've defined Ψ_T, let's get rid of it. If we integrate the equations for the amount of mass that gets locked up in metals and remnants,

$$\int_0^t \frac{d\mathcal{M}_s}{dt} = \int_0^t (1-R-D)\Psi \implies \mathcal{M}_s = (1-R-D)\Psi_T \qquad (32.29)$$

$$\int_0^t \frac{d\mathcal{M}_w}{dt} = D\Psi \implies \mathcal{M}_w = D\,\Psi_T \qquad (32.30)$$

So

$$\begin{aligned}\bar{Z}_s &= \frac{(1-R)y_z\Psi_T}{(1-R)\Psi_T} - \frac{\mathcal{M}_g Z}{\mathcal{M}_s + \mathcal{M}_w} \\ &= y_z - \frac{\mathcal{M}_g Z}{\mathcal{M}_s + \mathcal{M}_w} \qquad (32.31)\end{aligned}$$

Now let's define μ as the gas fraction of a galaxy

$$\mu = \frac{\mathcal{M}_g}{\mathcal{M}_g + \mathcal{M}_s + \mathcal{M}_w} = \frac{\mathcal{M}_g}{\mathcal{M}_t} \tag{32.32}$$

So

$$\bar{Z}_s = y_z - \frac{\mathcal{M}_t \mu}{\mathcal{M}_t - \mathcal{M}_t \mu} Z = y_z - \left(\frac{\mu}{1-\mu} \right) Z \tag{32.33}$$

So, if you choose a region (say, the solar neighborhood), and measure the average stellar metallicity, the present metal abundance in the interstellar medium, and the gas fraction, you can estimate the stellar yield of the elements in question. This is a key constraint on our understanding of the creation of metals by supernovae.

Probing Star Formation

The above equations allow us to estimate the history of star formation in the solar neighborhood. For example, if we re-write (32.18) as

$$\Psi = \frac{1}{1-R-D}\frac{dM_s}{dt} \tag{32.34}$$

and split the derivative into three parts, we get

$$\Psi = \frac{1}{1-R-D}\left(\frac{d\log\mathcal{M}_s}{d\log Z}\right)\left(\frac{d\log Z}{dt}\right)\left(\frac{d\mathcal{M}_s}{d\log\mathcal{M}_s}\right) \tag{32.35}$$

Now consider the derivatives. Since we can measure the metallicity of solar neighborhood stars, we can determining how much stellar mass there is as a function of metallicity. Thus the first derivative in (32.35) is a measureable quantity. Similarly, if we study nearby F-stars, and compare their absolute luminosities to the F-star zero-age main sequence luminosity, we can estimate how much main-sequence evolution has occurred, *i.e.,* we can estimate their ages. If we measure the stars' metallicities as well, then we have $d\log Z/dt$. Finally, the last derivative is simply $\ln(10)\mathcal{M}_s$. Thus, we can measure the history of star formation in the solar neighborhood.

This leads to the long-standing G-dwarf problem. Simple models of chemical evolution (say, a closed-box model or a constant infall model) predict many more low-metallicity stars in the solar neighborhood than are observed.

To get the history of matter infall, we can do a similar manipulation with equation (32.19)

$$\begin{aligned} f &= \frac{d\mathcal{M}_g}{dt} + (1-R)\Psi \\ &= \frac{d\mathcal{M}_g}{d\log Z} + \left\{ \frac{1-R}{1-R-D} \frac{d\mathcal{M}_s}{d\log Z} \right\} \frac{d\log Z}{dt} \end{aligned} \qquad (32.36)$$

If we measure the metallicities of clouds of H I and H_2 of different masses, then all terms of this equation are known, and the history of matter infall can be found.

The Closed Box Model of Chemical Evolution

As an example of what a chemical evolution model can do, consider a closed system, where all the material for current star formation comes from mass lost by a previous generation of stars. In this case, there is no infall, and, from (32.23),

$$\mathcal{M}_g \frac{dZ}{dt} = y_z \Psi(1-R) + (Z_f - Z) f = y_z \Psi(1-R) \tag{32.37}$$

In addition, from (32.19), we have

$$\frac{d\mathcal{M}_g}{dt} = -(1-R)\Psi + f = -(1-R)\Psi \tag{32.38}$$

By dividing these two equations, we get

$$\mathcal{M}_g \frac{dZ}{dt} \Bigg/ \frac{d\mathcal{M}_g}{dt} = \mathcal{M}_g \, \frac{dZ}{d\mathcal{M}_g} = -y_z \tag{32.39}$$

Since y_z is a constant of stellar evolution

$$\int_{Z_0}^{Z_1} dZ = -y_z \int_{\mathcal{M}_{g_0}}^{\mathcal{M}_{g_1}} \frac{d\mathcal{M}_g}{\mathcal{M}_g} \Longrightarrow \; Z_1 - Z_0 = -y_z \ln\left(\frac{\mathcal{M}_{g_0}}{\mathcal{M}_{g_1}}\right) \tag{32.40}$$

where Z_0 and $\mathcal{M}_{g_0}$ represent the initial metallicity and gas mass of the galaxy, and Z_1 and $\mathcal{M}_{g_1}$ represent those quantities today. Now let

$$\mu = \left(\frac{\mathcal{M}_g}{\mathcal{M}_t}\right) \quad \sigma = \left(\frac{\mathcal{M}_s}{\mathcal{M}_t}\right) \quad \delta = \left(\frac{\mathcal{M}_D}{\mathcal{M}_t}\right) \tag{32.41}$$

so

$$Z_1 - Z_0 = -y_z \ln\left(\frac{\mu_1}{\mu_0}\right) \tag{32.42}$$

or

$$\mu_1 = \mu_0 \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{32.43}$$

In other words, as the system evolves, the gas fraction will decrease exponentially with Z. Of course, we can't measure the metallicity evolution of the ISM directly, but we can use stars as a probe. If we take the derivative of (32.43) with respect to Z, then

$$\frac{d\mu}{dZ} = -\frac{\mu_1}{y_z} \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{32.44}$$

Meanwhile, through (32.18) and (32.19)

$$\frac{d\mathcal{M}_s}{dt} \Bigg/ \frac{d\mathcal{M}_g}{dt} = \frac{d\sigma}{d\mu} = -\frac{(1-R-D)}{(1-R)} \tag{32.45}$$

so

$$\frac{d\sigma}{dZ} = \left(\frac{d\mu}{dZ}\right)\left(\frac{d\sigma}{d\mu}\right) = \left(\frac{\mu}{y_z}\right)\left(\frac{1-R-D}{1-R}\right) \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{32.46}$$

Finally, if we put this equation in terms of $\log Z$, instead of Z, then

$$\frac{d\sigma/\sigma_1}{d\log Z} = (\ln 10)\left(\frac{Z_0}{y_z}\right)\left(\frac{1-R-D}{1-R}\right)\left(\frac{\mu_1}{\sigma_1}\right) \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{32.47}$$

Because the number of stellar metallicity measurements has (traditionally) not been extremely large, many times people plot the cumulative distribution, *i.e.,* the number of stars with metallicities less than Z. This is simply found by integrating (32.47). If we collect the constant terms and let

$$G = \left(\frac{1-R-D}{1-R}\right)\left(\frac{\mu_1}{\sigma_1}\right) \tag{32.48}$$

then for the cumulative distribution

$$\frac{\sigma}{\sigma_1} = 1 - G\left\{\exp\left(-\frac{Z_1 - Z_0}{y_z}\right) - 1\right\} \tag{32.49}$$

When this is fit against observations, it is clear that the metallicities of stars in the solar neighborhood cannot be fit with a closed box model of galactic evolution. Either $f \neq 0$, or $Z_0 \neq 0$, or the initial population of stars did not have the same IMF as the stars today, or there are severe chemical inhomogeneities in the ISM, and star formation occurs preferentially in regions with high metallicity.

A similar closed-box calculation can be performed for secondary elements. For these, if you divide (32.27) by (32.19), then

$$\mathcal{M}_g \frac{dX}{dt} \Bigg/ \frac{d\mathcal{M}_g}{dt} = \mathcal{M}_g \frac{dX}{d\mathcal{M}_g} = \frac{\Psi(1-R)(Z-X)y_x}{-\Psi(1-R)} \tag{32.50}$$

If we assume that $X \ll Z$, then this simply reduces to

$$\mathcal{M}_g dX = -y_z Z d\mathcal{M}_g \tag{32.51}$$

Now if we assume that $\mathcal{M}_g = \mathcal{M}_t$ at $t = 0$, then from (32.42)

$$\mathcal{M}_g = \mathcal{M}_t \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} \tag{32.52}$$

or

$$d\mathcal{M}_g = -\left(\frac{\mathcal{M}_g}{y_z}\right) \exp\left\{-\frac{Z_1 - Z_0}{y_z}\right\} dZ = \left(\frac{\mathcal{M}_g}{y_z}\right) dZ \tag{32.53}$$

Thus

$$\mathcal{M}_g dX = \frac{y_x}{y_z} Z dZ \Longrightarrow X = \frac{1}{2}\left(\frac{y_x}{y_z}\right) Z^2 \tag{32.54}$$

In other words, if X is a secondary element, then a plot of log-X versus log-Z (*i.e.*, [X] vs. [Z]) should have a slope of 2 (and presumably go through the solar value).

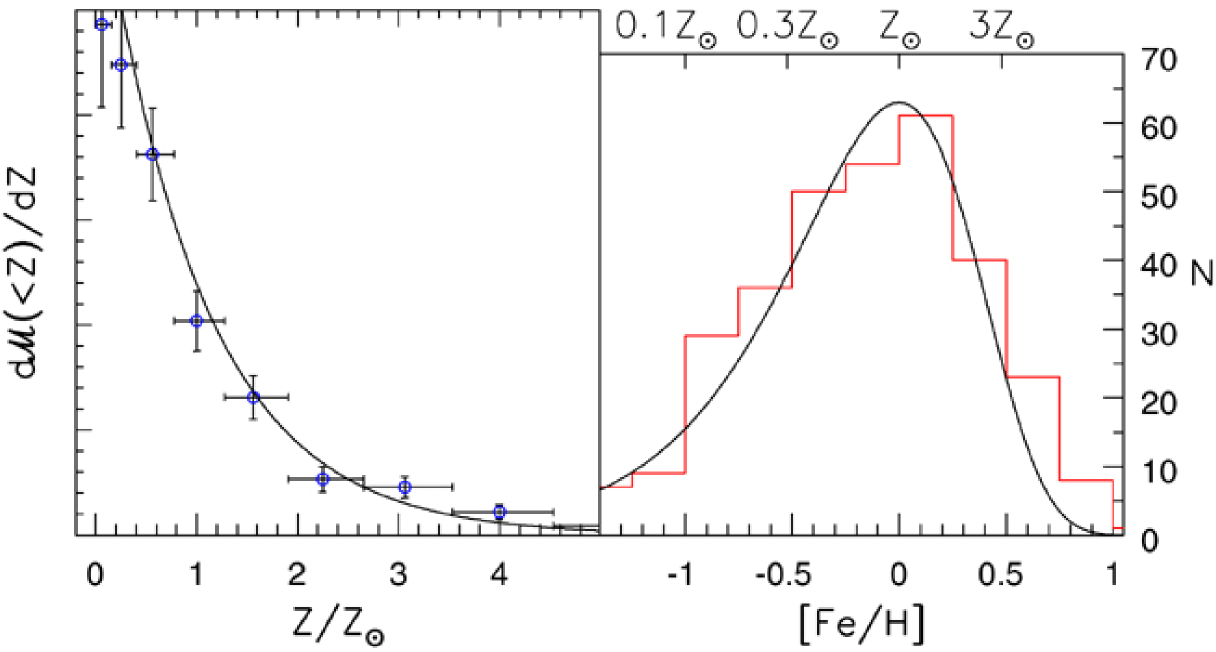


Fig 4.16 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Timescales for Chemical Evolution

The timescales for chemical evolution are simple to derive. First, let's consider the gas consumption timescale. In the absense of accretion, how long does it take to use up the gas?

$$\tau_* = \mathcal{M}_g \Bigg/ \left| \frac{d\mathcal{M}_g}{dt} \right| = \frac{\mathcal{M}_g}{-(1-R)\Psi + f} = \frac{\mathcal{M}_g}{|(1-R)\Psi|} \tag{32.55}$$

In the solar neighborhood, $\mathcal{M}_g \sim 5.7\mathcal{M}_\odot$ pc^{-2}, and the star formation rate is $\Psi \sim 4.2\mathcal{M}_\odot$ pc^{-2} Gyr^{-1}. For $R \sim 0.4$, that means that the gas should e-fold in $\tau_* \sim 2$ Gyr, much less than a Hubble time.

Similarly, we can calculate the timescale for chemical enrichment.

$$\begin{aligned} \tau_z = Z \Bigg/ \left| \frac{dZ}{dt} \right| &= \frac{Z}{|y_z\Psi(1-R) + (Z_f - Z)f| / \mathcal{M}_g} \\ &= \frac{\mathcal{M}_g Z}{|\Psi(1-R)y_z|} = \frac{\tau_* Z}{y_z} \end{aligned} \tag{32.56}$$

Since $y_z \sim Z$, $\tau_z \sim \tau_*$. In other words, for the solar neighborhood, the metallicity (and amount) of the gas should e-fold rather quickly. This strongly suggests that some our region of space has received matter from some other region.

The Balanced Infall Model

A closed-box model is not realistic for most systems: there is good evidence that infall of new material plays a part in the chemical evolution of a region. So let's calculate (for simplicity) a model where the infall just balances the amount of material becoming locked up into stars. In other words, a model where the mass in the interstellar medium stays constant. In this case

$$\frac{d\mathcal{M}_g}{dt} = -\Psi + E + f = -(1-R)\Psi + f = 0 \tag{32.57}$$

So

$$\Psi = \frac{f}{1-R} \tag{32.58}$$

The equation for metals is then

$$\mathcal{M}_g \frac{dZ}{dt} = \Psi(1-R)y_z + (Z_f - Z)f = f(y_z + Z_f - Z) \tag{32.59}$$

Also, since there is accretion

$$\frac{d\mathcal{M}_t}{dt} = f \tag{32.60}$$

So

$$\mathcal{M}_g \frac{dZ}{dt} \bigg/ \frac{d\mathcal{M}_t}{dt} = \mathcal{M}_g \frac{dZ}{d\mathcal{M}_t} = \frac{f(y_z + Z_f - Z)}{f} = y_z + Z_f - Z \tag{32.61}$$

If we integrate this

$$\int_{Z_0}^{Z_1} \frac{dZ}{y_z + Z_f - Z} = \int_{\mathcal{M}_0}^{\mathcal{M}} \frac{d\mathcal{M}_t}{\mathcal{M}_g} \tag{32.62}$$

then

$$\ln\left\{\frac{y_z + Z_f - Z_0}{y_z + Z_f - Z_1}\right\} = \frac{\mathcal{M} - \mathcal{M}_0}{\mathcal{M}_g} \tag{32.63}$$

Now let ν represent the total amount of mass accreted, scaled to the mass in the ISM, *i.e.*, $\nu = (\mathcal{M} - \mathcal{M}_0)/\mathcal{M}_g$. Then

$$Z_1 = (y_z + Z_f)(1 - e^{-\nu}) + Z_0 e^{-\nu} \tag{32.64}$$

If the galaxy began with $Z_0 \sim 0$ and $Z_f \sim 0$, then

$$Z_1 = y_z(1 - e^{-\nu}) \tag{32.65}$$

Note that if you also assume that the galaxy began as an entirely gaseous system, then

$$\nu = \frac{\mathcal{M} - \mathcal{M}_0}{\mathcal{M}_g} = \frac{\mathcal{M} - \mathcal{M}_0}{\mathcal{M}_0} = \mu^{-1} - 1 \tag{32.66}$$

Thus, as $\mu \to 0$, $\nu \to \infty$, and $Z_1 \to y_z$. In other words, the metallicity of the system asymptotes out at the value of the stellar yield.

The balanced infall model is similar to the closed-box model in that $Z \propto y_z$, so the metallicity of a system is actually measuring the stellar yields. Also, note that Z/y_z is not a strong function of the gas fraction, or Ψ/f. It depends mostly on the current properties of the system.